

A generalization of the Littlewood-Paley inequality for the fractional Laplacian $(-\Delta)^{\alpha/2}$

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Abstract

We prove a parabolic version of the Littlewood-Paley inequality for the fractional Laplacian $(-\Delta)^{\alpha/2}$, where $\alpha \in (0, 2)$.

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1 Introduction

Let $T_{2,t}$ be the semigroup corresponding to the heat equation $u_t = \Delta u$ (see (2.8)). The classical Littlewood-Paley inequality says for any $p \in (1, \infty)$ and $f \in L_p(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \left(\int_0^\infty |\nabla T_{2,t} f|^2 dt \right)^{p/2} dx \leq N(p) \|f\|_p^p. \quad (1.1)$$

In [5] and [7] Krylov extended (1.1) by proving the following parabolic version in which H is a Hilbert space.

Theorem 1.1 *Let H be a Hilbert space, $p \in [2, \infty)$, $-\infty \leq a < b \leq \infty$, $f \in L_p((a, b) \times \mathbb{R}^d, H)$. Then*

$$\int_{\mathbb{R}^d} \int_a^b \left(\int_a^t |\nabla T_{2,t-s} f|_H^2 ds \right)^{p/2} dt dx \leq N(p) \int_{\mathbb{R}^d} \int_a^b |f|_H^p dt dx. \quad (1.2)$$

Let $\alpha \in (0, 2)$. The main goal of this article is to prove (1.2) with $\partial_x^{\alpha/2}$ and $T_{\alpha,t}$ in place of ∇ and $T_{2,t}$ respectively, where $T_{\alpha,t}$ is the semigroup corresponding to the equation $u_t = -(-\Delta)^{\alpha/2} u$. That is, we prove

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Theorem 1.2 *Let H be a Hilbert space, $p \in [2, \infty)$, $-\infty \leq a < b \leq \infty$, and f be an H -valued function of (t, x) , then*

$$\int_{\mathbb{R}^d} \int_a^b \left[\int_a^t |\partial_x^{\alpha/2} T_{\alpha, t-s} f(s, \cdot)(x)|_H^2 ds \right]^{p/2} dx dt \leq N(\alpha, p) \int_{\mathbb{R}^d} \int_a^b |f|_H^p dt dx. \quad (1.3)$$

If $f(t, x) = f(x)$, then (1.3) easily leads to the Littlewood-Paley inequality (1.1) with $\partial_x^{\alpha/2}$ and $T_{\alpha, t}$ in place of ∇ and $T_{2, t}$ (see Remark 2.5).

Our motivation is as follows. For several decades, the fractional Laplacian and partial differential equations with the fractional Laplacian have been studied by many authors, see for instance [2] and [9]. Motivated by this, we were tempted to construct an L_p -theory of stochastic partial differential equations of the type

$$du = -(-\Delta)^{\alpha/2} u dt + \sum_{k=1}^{\infty} f^k dw_t^k, \quad u(0, x) = 0. \quad (1.4)$$

Here $f = (f^1, f^2, \dots)$ is an ℓ_2 -valued random function of (t, x) , and w_t^k are independent one-dimensional Wiener processes. It turns out that if $f = (f^1, f^2, \dots)$ satisfies certain measurability condition, the solution of this problem is given by

$$u(t, x) = \sum_{k=1}^{\infty} \int_0^t T_{\alpha, t-s} f^k(s, \cdot)(x) dw_s^k, \quad (1.5)$$

and by Burkholder-Davis-Gundy inequality (see [6]), we have

$$\mathbb{E} \int_0^T \|\partial_x^{\alpha/2} u(t, \cdot)\|_{L_p}^p dt \leq N(p) \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \left[\int_0^t |\partial_x^{\alpha/2} T_{\alpha, t-s} f(s, \cdot)(x)|_{\ell_2}^2 ds \right]^{p/2} dx dt. \quad (1.6)$$

Actually if f is not random, then the reverse inequality also holds. Thus to prove $\partial_x^{\alpha/2} u \in L_p$ and to get a legitimate start of the L_p -theory of SPDEs of type (1.4), one has to estimate the right-hand side of (1.6). Later, we will see that (1.3) implies that for any solution u of equation (1.4),

$$\mathbb{E} \int_0^T \|u(t, \cdot)\|_{H_p^{\alpha/2}}^p dt \leq N(\alpha, p, T) \mathbb{E} \int_0^T \|f\|_{\ell_2}^p ds, \quad (1.7)$$

where $\|u\|_{H_p^{\alpha/2}} := \|(1 - \Delta)^{\alpha/4} u\|_{L_p}$.

As usual \mathbb{R}^d stands for the Euclidean space of points $x = (x^1, \dots, x^d)$, $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ and $B_r := B_r(0)$. For $\beta \in (0, 1)$, and functions $u(x)$ we set

$$\nabla_x u = \left(\frac{\partial}{\partial x^1} u, \dots, \frac{\partial}{\partial x^d} u \right), \quad \partial_x^\beta u(x) = \mathcal{F}^{-1}(|\xi|^\beta \hat{u}(\xi))(x)$$

where $\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx$ is the Fourier transform of f . If we write $N = N(\dots)$, this means that the constant N depends only on what are in parenthesis.

2 Main Result

In this section we introduce a slightly extended version of Theorem 1.2. Fix $\alpha \in (0, 2)$ and let $p_\alpha(t, x) = p(t, x)$, where $t > 0$, denote the Fourier inverse transform of $e^{-(2\pi)^\alpha t |\xi|^\alpha}$, that is,

$$p(t, x) := \int_{\mathbb{R}^d} e^{i\xi \cdot x} e^{-(2\pi)^\alpha t |\xi|^\alpha} d\xi$$

and $p(x) := p(1, x)$. For a suitable function h and $t > 0$, define

$$T_t h(x) := (p(t, \cdot) * h(\cdot))(x) := \int_{\mathbb{R}^d} p(t, x - y) h(y) dy, \quad (2.8)$$

$$(-\Delta)^{\frac{\beta}{2}} h(x) := \partial_x^\beta h := \mathcal{F}^{-1}(|\xi|^\beta \mathcal{F}(h)(\xi))(x).$$

Then, for $\beta > 0$,

$$\begin{aligned} \partial_x^\beta T_t h(x) &= \mathcal{F}^{-1}(|\xi|^\beta e^{-(2\pi)^\alpha t |\xi|^\alpha} \hat{h}(\xi)) \\ &= \int_{\mathbb{R}^d} e^{i\xi \cdot x} |\xi|^\beta e^{-(2\pi)^\alpha t^{1/\alpha} |\xi|^\alpha} d\xi * h(x) \\ &= t^{-d/\alpha} \int_{\mathbb{R}^d} e^{i\xi \cdot t^{-1/\alpha} x} |t^{-1/\alpha} \xi|^\beta e^{-(2\pi)^\alpha |\xi|^\alpha} d\xi * h(x) \\ &= t^{-\beta/\alpha} \cdot t^{-d/\alpha} \phi_\beta(x/t^{1/\alpha}) * h(x), \end{aligned} \quad (2.9)$$

where

$$\phi_\beta(x) := \int_{\mathbb{R}^d} |\xi|^\beta e^{i\xi \cdot x} e^{-(2\pi)^\alpha |\xi|^\alpha} d\xi = (-\Delta)^{\frac{\beta}{2}} p(x).$$

The following two lemmas are crucial in this article and are proved in section 3.

Lemma 2.1 Denote $\hat{\phi}_\beta(\xi) = \mathcal{F}(\phi_\beta)(\xi)$, then there exists a constant $N = N(d, \alpha, \beta) > 0$ such that

$$|\hat{\phi}_\beta(\xi)| \leq N |\xi|^\beta, \quad |\xi| |\hat{\phi}(\xi)| \leq N,$$

$$|\phi_\beta(x)| \leq N \left(\frac{1}{|x|^{d+\beta}} \wedge 1 \right) \quad \text{and} \quad |\nabla \phi_\beta(x)| \leq N \left(\frac{1}{|x|^{d+1+\beta}} \wedge 1 \right).$$

Lemma 2.2 For each $\alpha \in (0, 2)$ and $\beta > 0$, there exists a continuously differentiable function $\bar{\phi}_\beta(\rho)$ defined on $[0, \infty)$ such that for some positive constant K which depends on d, α, β ,

$$|\phi_\beta(x)| + |\nabla \phi_\beta(x)| + |x| |\nabla \phi_\beta(x)| \leq \bar{\phi}_\beta(|x|), \quad \int_0^\infty |\bar{\phi}'_\beta(\rho)| d\rho \leq K,$$

$$\bar{\phi}_\beta(\infty) = 0, \quad \int_r^\infty |\bar{\phi}'_\beta(\rho)| \rho^d d\rho \leq \frac{K}{r^\beta} \quad \forall r \geq (10)^{-1/\alpha}.$$

To make our inequality slightly extended, we consider convolutions (see (2.9)) with more general functions. Let $\psi(x)$ be a $C^1(\mathbb{R}^d)$ function such that $|\hat{\psi}(\xi)| \leq K|\xi|^\nu$ for some $\nu > 0$, $|\xi|^\lambda |\hat{\psi}(\xi)| \leq K$ for some $\lambda > 0$, and assume that for some $\delta \geq \frac{\alpha}{2}$, there exists a continuously differentiable function $\bar{\psi}$ satisfying

$$|\psi(x)| + |\nabla \psi(x)| + |x| |\nabla \psi(x)| \leq \bar{\psi}(|x|), \quad \int_0^\infty |\bar{\psi}'(\rho)| d\rho \leq K, \quad \bar{\psi}(\infty) = 0$$

and

$$\int_r^\infty |\bar{\psi}'(\rho)| \rho^d d\rho \leq (K/r^\delta), \quad \forall r \geq (10)^{-1/\alpha}. \quad (2.10)$$

By Lemma 2.1 and Lemma 2.2, we know $\phi_{\alpha/2}$ satisfies all the above assumptions. Define

$$\Psi_t h(x) := t^{-d/\alpha} \psi(\cdot/t^{1/\alpha}) * h(\cdot)(x).$$

For $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$, $t > a \geq -\infty$, and $x \in \mathbb{R}^d$, we define

$$\mathcal{G}_a f(t, x) := \left[\int_a^t |\Psi_{t-s} f(s, \cdot)(x)|_H^2 \frac{ds}{t-s} \right]^{1/2}, \quad \mathcal{G} = \mathcal{G}_{-\infty}.$$

Here is our main result. The proof is given in section 5.

Theorem 2.3 *Let $p \in [2, \infty)$, $-\infty \leq a < b \leq \infty$ and $f \in C_0^\infty((a, b) \times \mathbb{R}^d, H)$. Then*

$$\int_{\mathbb{R}^d} \int_a^b [\mathcal{G}_a f(t, x)]^p dt dx \leq N \int_{\mathbb{R}^d} \int_a^b |f(t, x)|_H^p dt dx, \quad (2.11)$$

where the constant N depends only on $d, p, \alpha, \nu, \lambda, \delta$ and K .

Remark 2.4 *Take $\psi = \phi_{\alpha/2}$, $\nu = \delta = \alpha/2$, $\lambda = 1$, $a = 0$ and $b = T$, then (2.11) implies*

$$\int_{\mathbb{R}^d} \int_0^T \left[\int_0^t |\partial_x^{\alpha/2} T_{\alpha, t-s} f(s, \cdot)(x)|_H^2 ds \right]^{p/2} dt dx \leq N \int_{\mathbb{R}^d} \int_0^T |f(t, x)|_H^p dt dx. \quad (2.12)$$

Remark 2.5 *Note that inequality (2.1) with $\partial_x^{\alpha/2}$ and $T_{\alpha, t}$ in place of ∇ and $T_{2, t}$ is an easy consequence of (2.12). Indeed, take $T = 2$ and $f(t, x) = f(x)$. The left-hand side of (2.12) is not less than*

$$\int_{\mathbb{R}^d} \int_1^2 \left[\int_0^1 |\partial_x^{\alpha/2} T_{\alpha, s} f(x)|_H^2 ds \right]^{p/2} dt dx = \int_{\mathbb{R}^d} \left[\int_0^1 |\partial_x^{\alpha/2} T_{\alpha, s} f(x)|_H^2 ds \right]^{p/2} dx.$$

Thus it follows that

$$\int_{\mathbb{R}^d} \left[\int_0^1 |\partial_x^{\alpha/2} T_{\alpha, s} f(x)|_H^2 ds \right]^{p/2} dx \leq N \int_{\mathbb{R}^d} \|f\|_H^p dx,$$

and the self-similarity $(\partial_x^{\alpha/2} T_{\alpha, s} f(c \cdot))(x) = c^{\alpha/2} (\partial_x^{\alpha/2} T_{\alpha, c^\alpha s} f)(cx)$ allows one to replace the upper limit 1 by infinity with the same constant N .

3 Preliminary estimates on $(-\Delta)^{\beta/2}p(t, x)$

In this section we study the upper bound of $|(-\Delta)^{\beta/2}p(t, x)|$ and $|\nabla(-\Delta)^{\beta/2}p(t, x)|$, and then we prove Lemma 2.1 and Lemma 2.2. Actually the arguments in this section allow one to get the upper bound of $|D^m(-\Delta)^{\beta/2}p(t, x)|$ for any $m \geq 0$.

Lemma 3.1 *There exists a constant $N = N(d, \alpha, \beta) > 0$ such that*

$$|(-\Delta)^{\frac{\beta}{2}}p(x)| \leq \frac{N}{|x|^{d+\beta}}. \quad (3.13)$$

Proof. See [3] for $d = 1$ and [4] for $d \geq 2$. Actually in [3], (3.13) is given only for $\beta = 0$. Also in [4], $(-\Delta)^{\frac{\beta}{2}}p(x)$ is estimated in terms of power series (Proposition 2.2), however the series does not converge if $\alpha > 1$. For these reasons, we give a detailed proof. Also some inequalities obtained in this proof will be used in the proof of Lemma 3.4.

For $d = 1$, since $|\xi|$ is an even function, we have

$$\begin{aligned} (-\Delta)^{\frac{\beta}{2}}p(x) &= \int_{\mathbb{R}} |\xi|^{\beta} e^{i\xi x} e^{-(2\pi)^{\alpha}|\xi|^{\alpha}} d\xi \\ &= 2\operatorname{Re} \int_0^{\infty} \xi^{\beta} e^{i\xi x} e^{-(2\pi)^{\alpha}\xi^{\alpha}} d\xi \\ &= 2\operatorname{Re} \frac{1}{x^{1+\beta}} \int_0^{\infty} \xi^{\beta} e^{i\xi} e^{-(2\pi)^{\alpha}(\xi/x)^{\alpha}} d\xi. \end{aligned} \quad (3.14)$$

Assume $0 < \alpha \leq 1$. Consider the integrand as a function of the complex variable ξ . Since the integrand in (3.14) is analytic in the complement of the non-positive real half line and is continuous at zero, if we take principal branch cut, for $N > 0$, the path integration is zero on the closed path

$$\gamma_N(t) := \begin{cases} t & \text{if } 0 \leq t \leq N \\ N + i(t - N) & \text{if } N \leq t \leq 2N \\ 3N - t + iN & \text{if } 2N \leq t \leq 3N \\ i(4N - t) & \text{if } 3N \leq t \leq 4N. \end{cases}$$

By letting $N \rightarrow \infty$, one can move the path of integration to the positive imaginary axis, and gets (note $|e^{-(2\pi)^{\alpha}(i\xi/x)^{\alpha}}| \leq 1$)

$$|(-\Delta)^{\frac{\beta}{2}}p(x)| = \left| 2\operatorname{Re} \frac{1}{x^{1+\beta}} \int_0^{\infty} (i\xi)^{\beta} e^{-\xi} e^{-(2\pi)^{\alpha}(i\xi/x)^{\alpha}} i d\xi \right| \leq \frac{2}{|x|^{1+\beta}} \int_0^{\infty} \xi^{\beta} e^{-\xi} d\xi \leq \frac{N}{|x|^{1+\beta}}.$$

If $1 < \alpha < 2$, we use another closed path

$$\gamma_N(t) := \begin{cases} t & \text{if } 0 \leq t \leq N \cos \frac{\pi}{2\alpha} \\ N \cos \frac{\pi}{2\alpha} + i \sin \frac{\pi}{2\alpha} \left(\frac{t}{\cos \frac{\pi}{2\alpha}} - N \right) & \text{if } N \cos \frac{\pi}{2\alpha} \leq t \leq 2N \cos \frac{\pi}{2\alpha} \\ (3N - \frac{t}{\cos \frac{\pi}{2\alpha}}) e^{i \frac{\pi}{2\alpha}} & \text{if } 2N \cos \frac{\pi}{2\alpha} \leq t \leq 3N \cos \frac{\pi}{2\alpha}. \end{cases}$$

Thanks to the path integration on the above path, which looks like formally replacing ξ by $\xi e^{i\frac{\pi}{2\alpha}}$, we get (since $|e^{-(2\pi)^\alpha(\xi e^{i\frac{\pi}{2\alpha}}/x)^\alpha}| = 1$)

$$\begin{aligned} |(-\Delta)^{\frac{\beta}{2}}p(x)| &\leq |2\operatorname{Re}\frac{1}{x^{1+\beta}}\int_0^\infty (\xi e^{i\frac{\pi}{2\alpha}})^\beta e^{i\xi e^{i\frac{\pi}{2\alpha}}} e^{-(2\pi)^\alpha(\xi e^{i\frac{\pi}{2\alpha}}/x)^\alpha} e^{i\frac{\pi}{2\alpha}} d\xi| \\ &\leq \frac{2}{|x|^{1+\beta}}\int_0^\infty \xi^\beta e^{-\xi \sin \frac{\pi}{2\alpha}} d\xi \leq \frac{N}{|x|^{1+\beta}}. \end{aligned}$$

Next, let $d \geq 2$. Since the function $(-\Delta)^{\frac{\beta}{2}}p(x)$ is radial, we may assume $x = (|x|, \dots, 0)$, and if we denote the surface of the d -dimensional unit ball by S^{d-1} and the surface measure by $d\sigma$, then from the spherical coordinate we have

$$\begin{aligned} (-\Delta)^{\frac{\beta}{2}}p(x) &= \int_{\mathbb{R}^d} |\xi|^\beta e^{i\xi^1|x|} e^{-(2\pi)^\alpha|\xi|^\alpha} d\xi \\ &= \int_{\mathbb{R}^d} |\xi|^\beta \cos(\xi^1|x|) e^{-(2\pi)^\alpha|\xi|^\alpha} d\xi \\ &= \int_0^\infty r^{\beta+d-1} \int_{S^{d-1}} \cos(r\sigma^1|x|) e^{-(2\pi)^\alpha|r|^\alpha} d\sigma dr. \end{aligned}$$

Furthermore we can express $\sigma \in S^{d-1}$ as $\sigma = (\cos \theta, \phi \sin \theta)$ with $\theta \in [0, \pi]$ and $\phi \in S^{d-2}$, and get

$$\begin{aligned} (-\Delta)^{\frac{\beta}{2}}p(x) &= \int_0^\infty r^{\beta+d-1} \int_0^\pi \sin^{d-2}(\theta) \int_{S^{d-2}} \cos(r \cos \theta |x|) e^{-(2\pi)^\alpha|r|^\alpha} d\phi d\theta dr \\ &= A_{d-2} \int_0^\infty r^{\beta+d-1} \int_0^\pi \sin^{d-2}(\theta) \cos(r \cos \theta |x|) e^{-(2\pi)^\alpha|r|^\alpha} d\theta dr, \end{aligned}$$

where A_{d-2} is the area of S_{d-2} and $A_0 := 1$. By the changes of variables $r|x| \rightarrow r$ and $t = \cos \theta$,

$$\begin{aligned} (-\Delta)^{\frac{\beta}{2}}p(x) &= A_{d-2} \frac{1}{|x|^{\beta+d}} \int_0^\infty r^{\beta+d-1} \int_0^\pi \sin^{d-2}(\theta) \cos(r \cos \theta) e^{-(2\pi)^\alpha(r/|x|)^\alpha} d\theta dr \\ &= A_{d-2} \frac{1}{|x|^{\beta+d}} \int_0^\infty r^{\beta+d-1} \int_{-1}^1 \cos(rt) e^{-(2\pi)^\alpha(r/|x|)^\alpha} (1-t^2)^{(d-3)/2} dt dr. \quad (3.15) \end{aligned}$$

To proceed further, we use Bessel function $J_n(z)$ and Whittaker function $W_{0,n}(z)$. For any complex z that is not negative real and any real $n > -\frac{1}{2}$, define

$$\begin{aligned} J_n(z) &:= \frac{(\frac{1}{2}z)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}} \int_{-1}^1 (1-t^2)^{n-1/2} \cos(zt) dt, \\ W_{0,n}(z) &:= \frac{e^{-z/2}}{\Gamma(n+\frac{1}{2})} \int_0^\infty [t(1+t/z)]^{n-1/2} e^{-t} dt \end{aligned} \quad (3.16)$$

where $\arg z$ is understood to take its principle value, that is, $|\arg z| < \pi$. It is known (see, for instance, [12] p.346, p.360 and [11] p.314) that the two functions are related by the formula

$$J_n(z) = \frac{1}{\sqrt{2\pi z}} \left(\exp\left\{\frac{1}{2}(n+\frac{1}{2})\pi i\right\} W_{0,n}(2iz) + \exp\left\{-\frac{1}{2}(n+\frac{1}{2})\pi i\right\} W_{0,n}(-2iz) \right).$$

In particular, if z is a positive real number,

$$J_n(z) = 2\operatorname{Re} \left[\frac{1}{\sqrt{2\pi z}} \exp\left\{\frac{1}{2}\left(n + \frac{1}{2}\right)\pi i\right\} W_{0,n}(2iz) \right]. \quad (3.17)$$

We also know (see, for instance, [12] p. 343)

$$W_{0,n}(z) = e^{-\frac{1}{2}z} \{1 + O(z^{-1})\}. \quad (3.18)$$

Due to (3.16) and (3.17), from (3.15) we have

$$\begin{aligned} & (-\Delta)^{\frac{\beta}{2}} p(x) \\ &= \frac{A_{d-2}}{|x|^{\beta+d}} \int_0^\infty r^{\beta+d-1} \int_{-1}^1 \cos(rt) e^{-(2\pi)^\alpha (r/|x|)^\alpha} (1-t^2)^{(d-3)/2} dt dr \\ &= \frac{A_{d-2}}{|x|^{\beta+d}} \int_0^\infty r^{\beta+d/2} 2^{d/2-1} \Gamma\left(\frac{1}{2}(d-1)\right) \sqrt{\pi} J_{(d/2)-1}(r) e^{-(2\pi)^\alpha (r/|x|)^\alpha} dr \end{aligned} \quad (3.19)$$

$$= \frac{N(d)}{|x|^{\beta+d}} \operatorname{Re} \int_0^\infty r^{\beta+(d-1)/2} \exp\left\{\frac{1}{2}\left(\frac{d}{2} - \frac{1}{2}\right)\pi i\right\} W_{0,(d/2)-1}(2ir) e^{-(2\pi)^\alpha (r/|x|)^\alpha} dr, \quad (3.20)$$

where $N(d) := 2^{(d-1)/2} A_{d-2} \Gamma(\frac{1}{2}(d-1))$. From definition (3.16) one easily checks that the integrand in (3.20) is analytic in the complement of the non-positive real half line and is continuous at zero.

Let $0 < \alpha \leq 1$. Remembering (3.18) and doing the path integration on an appropriate closed path, as in the case $d = 1$, we can change the path of integration in (3.20) from the positive real half line to the negative imaginary half line. Taking this new path of integration, that is to say, formally replacing r by $-ir$, one gets (note $|e^{-(2\pi)^\alpha (-ir/|x|)^\alpha}| \leq 1$)

$$\begin{aligned} & |(-\Delta)^{\frac{\beta}{2}} p(x)| \\ &= \frac{N(d)}{|x|^{\beta+d}} \left| \operatorname{Re} \int_0^\infty (-ir)^{\beta+(d-1)/2} \exp\left\{\frac{1}{2}\left(\frac{d}{2} - \frac{1}{2}\right)\pi i\right\} W_{0,(d/2)-1}(2r) e^{-(2\pi)^\alpha (-ir/|x|)^\alpha} i dr \right| \\ &\leq \frac{N}{|x|^{\beta+d}} \int_0^\infty r^{\beta+(d-1)/2} W_{0,(d/2)-1}(2r) dr \leq \frac{N}{|x|^{\beta+d}}. \end{aligned}$$

Let $1 < \alpha < 2$. Then $|e^{-ire^{-i\frac{\pi}{2\alpha}}}| \leq e^{\frac{-r}{2}}$, and thus

$$|W_{0,(d/2)-1}(2ire^{-i\frac{\pi}{2\alpha}})| \leq \frac{e^{\frac{-r}{2}}}{\Gamma(d/2 - 1/2)} \int_0^\infty \left| [t(1 + t/(2ire^{-i\frac{\pi}{2\alpha}}))]^{(d-3)/2} e^{-t} \right| dt.$$

Note that if $d \geq 3$ then

$$|1 + t/(2ire^{-i\frac{\pi}{2\alpha}})|^{(d-3)/2} \leq |1 + t/r|^{(d-3)/2},$$

and if $d = 2$ then

$$|1 + t/(2ire^{-i\frac{\pi}{2\alpha}})|^{-1/2} \leq (1 + t \sin \frac{\pi}{2\alpha} / (2r))^{-1/2} \leq 2(1 + t/r)^{-1/2}.$$

It follows that for any $r > 0$, we have $|W_{0,(d/2)-1}(2ire^{-i\frac{\pi}{2\alpha}})| \leq 2W_{0,(d/2)-1}(r)$.

We do the path integration on a different closed path and change the path of integration in (3.20) from the positive real half line to the half line $\{re^{-i\frac{\pi}{2\alpha}} : r > 0\}$. Taking this new path of integration, that is to say, formally replacing r by $re^{-i\frac{\pi}{2\alpha}}$, one gets (note $|e^{-(2\pi)^\alpha(re^{-i\frac{\pi}{2\alpha}}/|x|)^\alpha}| = 1$)

$$\begin{aligned} |(-\Delta)^{\frac{\beta}{2}}p(x)| &\leq \frac{N}{|x|^{\beta+d}} \int_0^\infty \left| (re^{-i\frac{\pi}{2\alpha}})^{\beta+(d-1)/2} W_{0,(d/2)-1}(2ire^{-i\frac{\pi}{2\alpha}}) e^{-(2\pi)^\alpha(re^{-i\frac{\pi}{2\alpha}}/|x|)^\alpha} \right| dr \\ &\leq \frac{N}{|x|^{\beta+d}} \int_0^\infty r^{\beta+\frac{d-1}{2}} W_{0,(d/2)-1}(r) dr \leq \frac{N}{|x|^{\beta+d}}. \end{aligned}$$

The lemma is proved. \square

Remark 3.2 In the proof of Lemma 3.1 (see (3.14) and (3.19)) we proved that for any $\beta \geq 0$,

$$\left| \int_0^\infty \xi^\beta e^{i\xi} e^{-(2\pi)^\alpha(\xi/x)^\alpha} d\xi \right| < N, \quad \text{when } d = 1, \quad (3.21)$$

$$\left| \int_0^\infty r^{\beta+d/2} J_{(d/2)-1}(r) e^{-(2\pi)^\alpha(r/|x|)^\alpha} dr \right| < N, \quad \text{when } d \geq 2, \quad (3.22)$$

where $N = N(\alpha, \beta, d) > 0$ is independent of x .

Remark 3.3 Even though (3.13) is enough for our need, we believe it is not sharp. Actually it is known (see [1]) that if $\beta = 0$, then

$$p(t, x) \sim \left(\frac{t}{|x|^{d+\alpha}} \wedge t^{-d/\alpha} \right).$$

Lemma 3.4 There exists a constant $N = N(d, \alpha, \beta) > 0$ such that

$$|\nabla(-\Delta)^{\frac{\beta}{2}}p(x)| \leq N \left(\frac{1}{|x|^{\beta+d+1}} \vee \frac{1}{|x|^{\beta+d+\alpha+1}} \right). \quad (3.23)$$

Proof. Let $d = 1$. By (3.21),

$$\begin{aligned} \left| \frac{d}{dx} (-\Delta)^{\frac{\beta}{2}}p(x) \right| &= \left| \int_{\mathbb{R}} i\xi |\xi|^\beta e^{i\xi x} e^{-(2\pi)^\alpha|\xi|^\alpha} d\xi \right| \\ &\leq \frac{1}{|x|^{\beta+2}} \left| \int_{\mathbb{R}} \xi |\xi|^\beta e^{i\xi} e^{-(2\pi)^\alpha|\xi/x|^\alpha} d\xi \right| \\ &= \frac{2}{|x|^{\beta+2}} \left| \text{Im} \int_0^\infty \xi^{1+\beta} e^{i\xi} e^{-(2\pi)^\alpha|\xi/x|^\alpha} d\xi \right| \leq \frac{N}{|x|^{\beta+2}}. \end{aligned}$$

Let $d \geq 2$. From (3.19) and the inequality

$$\left| \frac{\partial}{\partial x_i} (-\Delta)^{\frac{\beta}{2}}p(x) \right| = \left| \frac{\partial}{\partial |x|} (-\Delta)^{\frac{\beta}{2}}p(x) \frac{\partial |x|}{\partial x_i} \right| \leq \left| \frac{\partial}{\partial |x|} (-\Delta)^{\frac{\beta}{2}}p(x) \right|$$

it easily follows that

$$\begin{aligned}
& |(-\Delta)^{\frac{\beta}{2}} p(x)| \\
& \leq \frac{N_1}{|x|^{\beta+d+1}} \left| \int_0^\infty (r)^{\beta+d/2} J_{d/2-1}(r) e^{-(2\pi)^\alpha (r/|x|)^\alpha} dr \right| \\
& \quad + \frac{N_2}{|x|^{\beta+d+\alpha+1}} \left| \int_0^\infty (r)^{\beta+d/2+\alpha} J_{d/2-1}(r) e^{-(2\pi)^\alpha (r/|x|)^\alpha} dr \right|.
\end{aligned}$$

Thus by (3.22),

$$|(-\Delta)^{\frac{\beta}{2}} p(x)| \leq N \left(\frac{1}{|x|^{\beta+d+1}} \vee \frac{1}{|x|^{\beta+d+\alpha+1}} \right).$$

The lemma is proved. \square

(Proof of Lemma 2.1)

First two assertions come from the fact

$$\mathcal{F}(\phi_\beta(x))(\xi) = \mathcal{F}\left(\int_{\mathbb{R}^d} |\eta|^\beta e^{i\eta \cdot x} e^{-(2\pi)^\alpha |\eta|^\alpha} d\eta\right)(\xi) = |\xi|^\beta e^{-(2\pi)^\alpha |\xi|^\alpha}.$$

Next, observe that

$$|\phi_\beta(x)| = |(-\Delta)^{\frac{\beta}{2}} p(x)| = \left| \int_{\mathbb{R}^d} |\xi|^\beta e^{i\xi \cdot x} e^{-(2\pi)^\alpha |\xi|^\alpha} d\xi \right| \leq \int_{\mathbb{R}^d} |\xi|^\beta e^{-(2\pi)^\alpha |\xi|^\alpha} d\xi < \infty.$$

Similarly,

$$|\nabla \phi_\beta(x)| \leq \int_{\mathbb{R}^d} |\xi|^{\beta+1} e^{-(2\pi)^\alpha |\xi|^\alpha} d\xi < \infty.$$

Therefore, by Lemma 3.1 and Lemma 3.4, there exists a constant $N(d, \alpha, \beta) > 0$ such that

$$|\phi_\beta(x)| \leq N \left(\frac{1}{|x|^{d+\beta}} \wedge 1 \right), \quad |\nabla \phi_\beta(x)| \leq N \left(\frac{1}{|x|^{d+1+\beta}} \wedge 1 \right).$$

The lemma is proved. \square

(Proof of Lemma 2.2)

By the inequalities in Lemma 2.1, we have

$$|\phi_\beta(x)| + |\nabla \phi_\beta(x)| + |x| |\nabla \phi_\beta(x)| \leq N \left(\frac{1}{|x|^{d+\beta}} \wedge 1 \right).$$

Define

$$\bar{\phi}_\beta(\rho) = \begin{cases} \frac{N}{\rho^{d+\beta}} & \text{if } \rho \geq (10)^{-1/\alpha} \\ N \cdot (10)^{(d+\beta)/\alpha} e^{-(d+\beta)((10)^{1/\alpha} \rho - 1)} & \text{if } \rho < (10)^{-1/\alpha}. \end{cases}$$

Then, $\bar{\phi}_\beta$ is continuously differentiable on $[0, \infty)$ such that

$$\bar{\phi}_\beta(\infty) = 0, \quad |\phi(x)| + |\nabla \phi(x)| + |x| |\nabla \phi(x)| \leq \bar{\phi}_\beta(|x|), \quad \int_0^\infty |\bar{\phi}'_\beta(\rho)| d\rho \leq K$$

and for each $r \geq (10)^{-1/\alpha}$,

$$\int_r^\infty |\overline{\phi}'_\beta(\rho)| \rho^d d\rho = \int_r^\infty (d + \beta) \frac{N}{\rho^{d+1+\beta}} \rho^d d\rho = \frac{(d + \beta)N}{\beta} r^{-\beta}.$$

The lemma is proved. \square

4 Some estimates on $\mathcal{G}f$

In this section we develop some estimates of $\mathcal{G}f$ by adopting the approaches in [7], where the case $\alpha = 2$ is studied. Fix $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$ and denote $u = \mathcal{G}f$.

First, we prove a version of Theorem 2.3 when $p = 2$.

Lemma 4.1 *There exists a constant $N = N(\nu, \lambda, \alpha, K) > 0$ so that for any $T \in (-\infty, \infty]$,*

$$\|u\|_{L_2(\mathbb{R}^{d+1} \cap \{t \leq T\})}^2 \leq N \|f\|_{L_2(\mathbb{R}^{d+1} \cap \{t \leq T\})}^2. \quad (4.24)$$

Proof. By the continuity of f , the range of f belongs to a separable subspace of H . Thus by using a countable orthonormal basis of this subspace and the Fourier transform one easily finds

$$\begin{aligned} \|u\|_{L_2(\mathbb{R}^{d+1} \cap \{t \leq T\})}^2 &= \int_{\mathbb{R}^d} \int_{-\infty}^T \left[\int_{-\infty}^t |\hat{\psi}(\xi(t-s)^{1/\alpha})|^2 |\hat{f}(s, \xi)|_H^2 \frac{ds}{t-s} \right] dt d\xi \\ &= \int_{\mathbb{R}^d} \int_{-\infty}^T \int_{-\infty}^T I_{s \leq t} |\hat{\psi}(\xi(t-s)^{1/\alpha})|^2 |\hat{f}(s, \xi)|_H^2 \frac{dt}{t-s} ds d\xi \\ &= \int_{\mathbb{R}^d} \int_{-\infty}^T \int_0^{T-s} |\hat{\psi}(\xi t^{1/\alpha})|^2 \frac{dt}{t} |\hat{f}(s, \xi)|_H^2 ds d\xi. \end{aligned} \quad (4.25)$$

By the assumption on ψ , for some $\nu, \lambda, K > 0$,

$$|\hat{\psi}(\xi)| \leq K |\xi|^\nu, \quad |\xi|^\lambda |\hat{\psi}(\xi)| \leq K.$$

This and the change of the variables $|\xi|^\alpha t \rightarrow t$ easily lead to

$$\begin{aligned} &\int_0^\infty |\hat{\psi}(\xi t^{1/\alpha})|^2 \frac{dt}{t} = \int_0^\infty |\hat{\psi}(t^{1/\alpha} \frac{\xi}{|\xi|})|^2 \frac{dt}{t} \\ &\leq K^2 \int_0^1 t^{-1+2\nu/\alpha} dt + K^2 \int_1^\infty t^{-1-2\lambda/\alpha} dt \leq N(\nu, \alpha, \lambda, K). \end{aligned} \quad (4.26)$$

Plugging (4.26) into (4.25),

$$\|u\|_{L_2(\mathbb{R}^{d+1} \cap \{t \leq T\})}^2 \leq N \int_{-\infty}^T \int_{\mathbb{R}^d} |\hat{f}(s, \xi)|_H^2 d\xi ds.$$

The last expression is equal to the right-hand side of (4.24), and therefore the lemma is proved. \square

For a real-valued function h defined on \mathbb{R}^d , define the maximal function

$$\mathbb{M}_x h(x) := \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |h(y)| dy,$$

where $|B_r(x)|$ denotes Lebesgue measure of $B_r(x)$. Similarly, for measurable functions $h = h(t)$ on \mathbb{R} we introduce $\mathbb{M}_t h$ as the maximal function of h relative to symmetric intervals:

$$\mathbb{M}_t h(t) := \sup_{r>0} \frac{1}{2r} \int_{-r}^r |h(t+s)| ds.$$

For a function $h(t, x)$ of two variables, set

$$\mathbb{M}_x h(t, x) := \mathbb{M}_x(h(t, \cdot))(x), \quad \mathbb{M}_t h(t, x) = \mathbb{M}_t(h(\cdot, x))(t).$$

Denote

$$Q_0 := [-2^\alpha, 0] \times [-1, 1]^d. \quad (4.27)$$

Corollary 4.2 *Assume that the support of f is within $[-10, 10] \times B_{3d}$. Then for any $(t, x) \in Q_0$*

$$\int_{Q_0} |u(s, y)|^2 ds dy \leq N \mathbb{M}_t \mathbb{M}_x |f|_H^2(t, x), \quad (4.28)$$

where N depends only on d, α, ν, λ and K .

Proof. By the Lemma 4.1,

$$\int_{Q_0} |u(s, y)|^2 ds dy \leq \int_{-\infty}^0 \int_{\mathbb{R}^d} |u(s, y)|^2 dy ds \leq N \int_{-10}^0 \int_{B_{3d}} |f(s, y)|_H^2 dy ds.$$

Since $|x - y| \leq |x| + |y| \leq 4d$ for any $(t, x) \in Q_0$ and $y \in B_{3d}$,

$$\begin{aligned} \int_{-10}^0 \int_{B_{3d}} |f(s, y)|_H^2 dy ds &\leq \int_{-10}^0 \int_{|x-y| \leq 4d} |f(s, y)|_H^2 dy ds \leq N \int_{-10}^0 \mathbb{M}_x |f(s, x)|_H^2 ds \\ &\leq N \mathbb{M}_t \mathbb{M}_x |f|_H^2(t, x). \end{aligned}$$

The lemma is proved. □

We generalize Corollary 4.2 as follows.

Lemma 4.3 *Assume that $f(t, x) = 0$ for $t \neq (-10, 10)$. Then for any $(t, x) \in Q_0$,*

$$\int_{Q_0} |u(s, y)|^2 ds dy \leq N \mathbb{M}_t \mathbb{M}_x |f|_H^2(t, x),$$

where $N = N(d, \alpha, \nu, \lambda, \delta, K)$.

Proof. First, notice that if $0 \leq \varepsilon \leq R \leq \infty$, and F and G are smooth enough, then

$$\begin{aligned} \int_{R \geq |z| \geq \varepsilon} F(z)G(|z|) dz &= - \int_{\varepsilon}^R G'(\rho) \left(\int_{|z| \leq \rho} F(z) dz \right) d\rho \\ &\quad + G(R) \int_{|z| \leq R} F(z) dz - G(\varepsilon) \int_{|z| \leq \varepsilon} F(z) dz. \end{aligned} \quad (4.29)$$

Indeed, (4.29) is obtained by applying integration by parts to

$$\int_{\varepsilon}^R G(\rho) \frac{d}{d\rho} \left(\int_{B_{\rho}(0)} F(z) dz \right) d\rho = \int_{\varepsilon}^R G(\rho) \left(\int_{\partial B_{\rho}(0)} F(s) dS_{\rho} \right) d\rho = \int_{R \geq |z| \geq \varepsilon} F(z)G(|z|) dz.$$

Now take $\zeta \in C_0^{\infty}(\mathbb{R}^d)$ such that $\zeta = 1$ in B_{2d} and $\zeta = 0$ outside of B_{3d} . Set $\mathcal{A} = \zeta f$ and $\mathcal{B} = (1 - \zeta)f$. By Minkowski's inequality, $\mathcal{G}f \leq \mathcal{G}\mathcal{A} + \mathcal{G}\mathcal{B}$. Since $\mathcal{G}\mathcal{A}$ can be estimated by Corollary 4.2, we may assume that $f(t, x) = 0$ for $x \in B_{2d}$.

Denote $\bar{f} = |f|_H$, take $0 > s > r > -10$, and see

$$\begin{aligned} |\Psi_{s-r} f(r, \cdot)(y)|_H &\leq (s-r)^{-d/\alpha} \int_{\mathbb{R}^d} |\psi(z/(s-r)^{1/\alpha})| |f(r, y-z)|_H dz \\ &\leq (s-r)^{-d/\alpha} \int_{\mathbb{R}^d} \bar{\psi}(|z|/(s-r)^{1/\alpha}) \bar{f}(r, y-z) dz. \end{aligned}$$

Observe that if $(s, y) \in Q_0$ and $|z| \leq \rho$ with a $\rho > 1$, then

$$|x - y| \leq 2d, \quad B_{\rho}(y) \subset B_{2d+\rho}(x) \subset B_{\mu\rho}(x), \quad \mu = 2d + 1, \quad (4.30)$$

whereas if $|z| \leq 1$, then $|y - z| \leq 2d$ and $f(r, y - z) = 0$. Thus by (4.29), for $0 > s > r > -10$ and $(s, y) \in Q_0$

$$\begin{aligned} |\Psi_{s-r} f(r, \cdot)(y)|_H &\leq (s-r)^{-(d+1)/\alpha} \int_1^{\infty} |\bar{\psi}'(\rho/(s-r)^{1/\alpha})| \left(\int_{|z| \leq \rho} \bar{f}(r, y-z) dz \right) d\rho \\ &= (s-r)^{-(d+1)/\alpha} \int_1^{\infty} |\bar{\psi}'(\rho/(s-r)^{1/\alpha})| \left(\int_{B_{\rho}(y)} \bar{f}(r, z) dz \right) d\rho \\ &\leq (s-r)^{-(d+1)/\alpha} \int_1^{\infty} |\bar{\psi}'(\rho/(s-r)^{1/\alpha})| \left(\int_{B_{\mu\rho}(x)} \bar{f}(r, z) dz \right) d\rho \\ &\leq N \mathbb{M}_x \bar{f}(r, x) (s-r)^{-(d+1)/\alpha} \int_1^{\infty} |\bar{\psi}'(\rho/(s-r)^{1/\alpha})| \rho^d d\rho \\ &= N \mathbb{M}_x \bar{f}(r, x) \int_{(s-r)^{-1/\alpha}}^{\infty} |\bar{\psi}'(\rho)| \rho^d d\rho \leq N \mathbb{M}_x \bar{f}(r, x) (s-r)^{\delta/\alpha}, \end{aligned}$$

where the last inequality follows from (2.10) and the inequality $(s-r)^{-1/\alpha} \geq 10^{-1/\alpha}$. By Jensen's inequality $(\mathbb{M}_x \bar{f})^2 \leq \mathbb{M}_x \bar{f}^2$, and therefore, for any $(s, y) \in Q_0$ (remember $\delta \geq \alpha/2$)

$$\begin{aligned} |u(s, y)|^2 &= \int_{-\infty}^s |\Psi_{s-r} f(r, \cdot)(y)|_H^2 \frac{dr}{s-r} \leq N \int_{-10}^s \mathbb{M}_x \bar{f}^2(r, x) (s-r)^{2\delta/\alpha-1} dr \\ &\leq N \int_{-10}^0 \mathbb{M}_x \bar{f}^2(r, x) dr \leq N \mathbb{M}_t \mathbb{M}_x \bar{f}^2(t, x). \end{aligned}$$

The lemma is proved. \square

Lemma 4.4 *Assume that $f(t, x) = 0$ for $t \geq -8$. Then for any $(t, x) \in Q_0$*

$$\int_{Q_0} |u(s, y) - u(t, x)|^2 ds dy \leq N \mathbb{M}_t \mathbb{M}_x |f|_H^2(t, x), \quad (4.31)$$

where $N = N(d, \alpha, \nu, \lambda, \delta, K)$.

Proof. Obviously it is enough to show that

$$\sup_{Q_0} [|D_s u|^2 + |\nabla u|^2] \leq N \mathbb{M}_t \mathbb{M}_x |f|_H^2(t, x). \quad (4.32)$$

By Minkowski's inequality the derivative of a norm is less than or equal to the norm of the derivative if both exist. Thus for fixed $(s, y) \in Q_0$ we have

$$|\nabla u(s, y)|^2 \leq \int_{-\infty}^{-8} |\nabla \Psi_{s-r} f(r, \cdot)(y)|_H^2 \frac{dr}{s-r} =: \int_{-\infty}^{-8} I^2(r, s, y) \frac{dr}{s-r},$$

where

$$\begin{aligned} I(r, s, y) &:= |\nabla \Psi_{s-r} f(r, \cdot)(y)|_H \\ &= (s-r)^{-(d+1)/\alpha} \left| \int_{\mathbb{R}^d} (\nabla \psi)(z/(s-r)^{1/\alpha}) f(r, y-z) dz \right|_H \\ &\leq (s-r)^{-(d+1)/\alpha} \int_{\mathbb{R}^d} \bar{\psi}(|z|/(s-r)^{1/\alpha}) \bar{f}(r, y-z) dz =: \tilde{I}(r, s, y), \end{aligned}$$

and $\bar{f} := |f|_H$. Using (4.29) and (4.30) again, we get for $s > r$,

$$\begin{aligned} \tilde{I}(r, s, y) &\leq (s-r)^{-(d+2)/\alpha} \int_0^\infty \bar{\psi}'(\rho/(s-r)^{1/\alpha}) \left(\int_{B_\rho(y)} \bar{f}(r, z) dz \right) d\rho \\ &\leq (s-r)^{-(d+2)/\alpha} \int_0^\infty \bar{\psi}'(\rho/(s-r)^{1/\alpha}) \left(\int_{B_{2d+\rho}(x)} \bar{f}(r, z) dz \right) d\rho \\ &\leq N \mathbb{M}_x \bar{f}(r, x) (s-r)^{-(d+2)/\alpha} \int_0^\infty \bar{\psi}'(\rho/(s-r)^{1/\alpha}) (2d+\rho)^d d\rho \\ &= N \mathbb{M}_x \bar{f}(r, x) (s-r)^{-1/\alpha} \int_0^\infty \bar{\psi}'(\rho) (2d/(s-r)^{1/\alpha} + \rho)^d d\rho. \end{aligned}$$

For $r \leq -8$, we have $s-r \geq 2^\alpha$ and

$$\int_0^\infty |\bar{\psi}'(\rho)| (2d/(s-r)^{1/\alpha} + \rho)^d d\rho \leq \int_0^\infty |\bar{\psi}'(\rho)| (d+\rho)^d d\rho \leq N,$$

$$\tilde{I}(r, s, y) \leq N \mathbb{M}_x \bar{f}(r, x) (s-r)^{-1/\alpha}$$

and

$$\begin{aligned} |\nabla u(s, y)|^2 &\leq \int_{-\infty}^{-8} \tilde{I}^2(r, s, y) \frac{dr}{s-r} \leq N \int_{-\infty}^{-8} \mathbb{M}_x \bar{f}^2(r, x) \frac{dr}{(s-r)^{2/\alpha+1}} \\ &\leq N \int_{-\infty}^{-8} \mathbb{M}_x \bar{f}^2(r, x) \frac{dr}{(-4-r)^{2/\alpha+1}}. \end{aligned}$$

By the integration by parts,

$$\begin{aligned} |\nabla u(s, y)|^2 &\leq N \int_{-\infty}^{-8} \tilde{I}^2(r, s, y) \frac{dr}{s-r} \\ &\leq N \int_{-\infty}^{-8} \frac{1}{(-4-r)^{2/\alpha+2}} \left(\int_r^0 \mathbb{M}_x \bar{f}^2(p, x) dp \right) dr \\ &\leq N \mathbb{M}_t \mathbb{M}_x \bar{f}^2(t, x) \int_{-\infty}^{-8} \frac{|r|}{(-4-r)^{2/\alpha+2}} dr = N \mathbb{M}_t \mathbb{M}_x \bar{f}^2(t, x). \end{aligned} \quad (4.33)$$

To estimate $D_s u$, we proceed similarly. By Minkowski's inequality,

$$\begin{aligned} |D_s u(s, y)|^2 &\leq N \int_{-\infty}^{-8} \left(|D_s \Psi_{s-r} f(r, y)|_H^2 \frac{1}{s-r} + |\Psi_{s-r} f(r, y)|_H^2 \frac{1}{(s-r)^3} \right) dr \\ &=: N \int_{-\infty}^{-8} J^2(r, s, y) \frac{1}{s-r} dr, \end{aligned} \quad (4.34)$$

where

$$\begin{aligned} J(r, s, y) &:= (s-r)^{-d/\alpha} \left| \int_{\mathbb{R}^d} D_s \psi(z/(s-r)^{1/\alpha}) f(r, y-z) dz \right|_H \\ &\quad + (s-r)^{-d/\alpha-1} \left| \int_{\mathbb{R}^d} \psi(z/(s-r)^{1/\alpha}) f(r, y-z) dz \right|_H \\ &= (s-r)^{-d/\alpha} \left| \int_{\mathbb{R}^d} \nabla \psi(z/(s-r)^{1/\alpha}) \cdot \left(-\frac{1}{\alpha} (s-r)^{-1/\alpha-1} z \right) f(r, y-z) dz \right|_H \\ &\quad + (s-r)^{-d/\alpha-1} \left| \int_{\mathbb{R}^d} \psi(z/(s-r)^{1/\alpha}) f(r, y-z) dz \right|_H \\ &\leq N (s-r)^{-d/\alpha-1} \int_{\mathbb{R}^d} \bar{\psi}(|z|/(s-r)^{1/\alpha}) \bar{f}(r, y-z) dz = N \tilde{I}(r, s, y). \end{aligned}$$

This, (4.33) and (4.31) lead to (5.1). The lemma is proved. \square

5 Proof of Theorem 2.3

Note that we may assume $a = -\infty$ and $b = \infty$. Indeed, for any $f \in C_0^\infty((a, b) \times \mathbb{R}^d, H)$ we have $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$, and inequality (2.11) with $a = -\infty$ and $b = \infty$ implies the inequality with any pair of (a, b) . Since in this case the theorem is already proved if $p = 2$, we assume $p > 2$.

Let \mathcal{F} be the collections of all balls $Q \subset \mathbb{R}^{d+1}$ of the type

$$Q_c(s, y) := \{(s - c^\alpha, s) \times (y^1 - c/2, y^1 + c/2) \cdots (y^d - c/2, y^d + c/2)\}, \quad c > 0.$$

For a measurable function $h(t, x)$ on \mathbb{R}^{d+1} , define the sharp function

$$h^\#(t, x) := \sup_Q \frac{1}{|Q|} \int_Q |h(t, x) - h_Q| dyds,$$

where

$$h_Q = \int_Q h dyds := \frac{1}{|Q|} \int_Q h(s, y) dyds$$

and the supremum is taken over all balls $Q \in \mathcal{F}$ containing (t, x) .

Theorem 5.1 (*Fefferman-Stein*). *For any $1 < q < \infty$ and $h \in L_q(\mathbb{R}^{d+1})$,*

$$\|h\|_{L^q} \leq N(q) \|h^\#\|_{L^q}. \quad (5.1)$$

Proof. Inequality (5.1) is a consequence of Theorem IV.2.2 in [10], because the balls $Q_c(s, y)$ satisfy the conditions (i)-(iv) in section 1.1 of [10] :

- (i) $Q_c(t, x) \cap Q_c(s, y) \neq \emptyset$ implies $Q_c(s, y) \subset Q_{N_1 c}(t, x)$;
- (ii) $|Q_{N_1 c}(t, x)| \leq N_2 |Q_c(t, x)|$;
- (iii) $\cap_{c>0} \overline{Q_c}(t, x) = \{(t, x)\}$ and $\cup_c Q_c(t, x) = \mathbb{R}^{d+1}$;
- (iv) for each open set U and $c > 0$, the function $(t, x) \rightarrow |Q_c(t, x) \cap U|$ is continuous. \square

Next we prove

$$(\mathcal{G}f)^\#(t, x) \leq N(\mathbb{M}_t \mathbb{M}_x |f|_H^2)^{1/2}(t, x). \quad (5.2)$$

By Jensen's inequality, to prove (5.2) it suffices to prove that for each $Q = Q_c(s, y) \in \mathcal{F}$ and $(t, x) \in Q$,

$$\int_Q |\mathcal{G}f - (\mathcal{G}f)_Q|^2 dyds \leq N(d, \alpha, \nu, \lambda, \delta, K) \mathbb{M}_t \mathbb{M}_x |f|_H^2(t, x). \quad (5.3)$$

It is easy to check that to prove (5.3) we may assume $(s, y) = (0, 0)$. Note that for any $c > 0$, $\Psi_t h(c \cdot)(x) = \Psi_{t c^\alpha} h(cx)$ and

$$\begin{aligned} \mathcal{G}f(c^\alpha \cdot, c \cdot)(t, x) &= \left[\int_{-\infty}^t |\Psi_{(t-s)c^\alpha} f(c^\alpha s, \cdot)(cx)|_H^2 \frac{ds}{t-s} \right]^{1/2} \\ &= \left[\int_{-\infty}^{t c^\alpha} |\Psi_{(t-c^{-\alpha}s)c^\alpha} f(s, \cdot)(cx)|_H^2 \frac{c^{-\alpha} ds}{t-c^{-\alpha}s} \right]^{1/2} \\ &= \left[\int_{-\infty}^{t c^\alpha} |\Psi_{(c^\alpha t-s)} f(s, \cdot)(cx)|_H^2 \frac{ds}{c^\alpha t-s} \right]^{1/2} \\ &= \mathcal{G}f(c^\alpha t, cx). \end{aligned} \quad (5.4)$$

Since dilations don't affect averages, (5.4) shows that it suffices to prove (5.3) when $c = 2$, that is $Q = Q_0$ from (4.27). Now we take a function $\zeta \in C_0^\infty(\mathbb{R})$ such that $\zeta = 1$ on $[-8, 8]$, $\zeta = 0$ outside of $[-10, 10]$, and $1 \geq \zeta \geq 0$. Define

$$\mathcal{A}(s, y) := f(s, y)\zeta(s), \quad \mathcal{B}(s, y) := f(s, y) - \mathcal{A}(s, y) = f(s, y)(1 - \zeta(s)).$$

Then

$$\Psi_{t-s}\mathcal{A}(s, \cdot) = \zeta(s)\Psi_{t-s}f(s, \cdot), \quad \mathcal{G}f \leq \mathcal{G}\mathcal{A} + \mathcal{G}\mathcal{B}, \quad \mathcal{G}\mathcal{B} \leq \mathcal{G}f$$

and for any constant c , $|\mathcal{G}f - c| \leq |\mathcal{G}\mathcal{A}| + |\mathcal{G}\mathcal{B} - c|$. Thus

$$\int_{Q_0} |\mathcal{G}f - (\mathcal{G}f)_{Q_0}|^2 dy ds \leq 4 \int_{Q_0} |\mathcal{G}f - c|^2 dy ds \leq 8 \int_{Q_0} |\mathcal{G}\mathcal{A}|^2 dy ds + 8 \int_{Q_0} |\mathcal{G}\mathcal{B} - c|^2 dy ds.$$

Taking $c = \mathcal{G}\mathcal{B}(t, x)$, from Lemma 4.3 we get

$$\begin{aligned} \int_{Q_0} |\mathcal{G}f - (\mathcal{G}f)_{Q_0}|^2 dy ds &\leq 8 \int_{Q_0} |\mathcal{G}\mathcal{A}|^2 dy ds + 8 \int_{Q_0} |\mathcal{G}\mathcal{B} - \mathcal{G}\mathcal{B}(t, x)|^2 dy ds \\ &\leq N \mathbb{M}_t \mathbb{M}_x |f|_H^2(t, x) + 8 \int_{Q_0} |\mathcal{G}\mathcal{B} - \mathcal{G}\mathcal{B}(t, x)|^2 dy ds. \end{aligned}$$

In addition, setting $f_1(s, y) := \mathcal{B}(s, y)$ on $s \leq 0$ and $f_1(s, y) := 0$ on $s > 0$, from Lemma 4.4 we see

$$\begin{aligned} \mathbb{M}_t \mathbb{M}_x |f|_H^2(t, x) + 8 \int_{Q_0} |\mathcal{G}\mathcal{B} - \mathcal{G}\mathcal{B}(t, x)|^2 dy ds &\leq \mathbb{M}_t \mathbb{M}_x |f|_H^2(t, x) + N \mathbb{M}_t \mathbb{M}_x |f_1|_H^2(t, x) \\ &\leq N \mathbb{M}_t \mathbb{M}_x |f|_H^2(t, x). \end{aligned}$$

This proves (5.2).

Finally, combining the Fefferman-Stein theorem and Hardy-Littlewood maximal theorem (see, for instance, [10]), we conclude (recall $p/2 > 1$)

$$\begin{aligned} \|u\|_{L_p(\mathbb{R}^{d+1})}^p &\leq N \|(\mathbb{M}_t \mathbb{M}_x |f|_H^2)^{1/2}\|_{L_p(\mathbb{R}^{d+1})}^p = N \int_{\mathbb{R}^d} \int_{\mathbb{R}} (\mathbb{M}_t \mathbb{M}_x |f|_H^2)^{p/2} dt dx \\ &\leq N \int_{\mathbb{R}^d} \int_{\mathbb{R}} (\mathbb{M}_x |f|_H^2)^{p/2} dt dx \\ &= N \int_{\mathbb{R}} \int_{\mathbb{R}^d} (\mathbb{M}_x |f|_H^2)^{p/2} dx dt \\ &\leq N \|f\|_{L_p(\mathbb{R}^{d+1}, H)}^p. \end{aligned}$$

The theorem is proved. □

Below we explain why Theorem 2.3 implies (1.7). By (1.6) and Remark 2.4, for any solution u of (1.4), we have

$$\mathbb{E} \int_0^T \|\partial_x^{\alpha/2} u(t, \cdot)\|_{L_p}^p dt \leq N \mathbb{E} \int_{\mathbb{R}^d} \int_0^T |f|_{\ell_2}^p dt dx. \quad (5.5)$$

By (1.5) and Burkholder-Davis-Gundy inequality,

$$\mathbb{E} \int_0^T \|u(t, \cdot)\|_{L_p}^p dt \leq N \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \left[\int_0^t |T_{\alpha, t-s} f(s, \cdot)(x)|_{\ell_2}^2 ds \right]^{p/2} dx dt, \quad (5.6)$$

and by Jensen's inequality

$$|T_{t-s} f(x)|_{\ell_2}^2 = \sum_k \left(\int_{\mathbb{R}^d} p(t-s, y) f^k(x-y) dy \right)^2 \leq N(p(t-s, \cdot) * |f(\cdot)|_{\ell_2}^2)(x). \quad (5.7)$$

Thus (5.5), (5.6), (5.7) and Remark 3.3 imply

$$\mathbb{E} \int_0^T \|u\|_{H_p^{\alpha/2}}^p dt \leq N \mathbb{E} \int_0^T (\|u\|_{L_p}^p + \|\partial^{\alpha/2} u\|_{L_p}^p) dt \leq N(T, d, \alpha) \mathbb{E} \int_{\mathbb{R}^d} \int_0^T |f|_{\ell_2}^p dt dx.$$

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